Exactly solved dynamics for an infinite-range spin system

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It is well known that the dynamical evolution of a system of N spins can be viewed as a walk along the edges of an N-dimensional hypercube. I use this correspondence in an infinite-range spin system to derive a diffusion equation for the magnetization. The diffusion equation then leads to an ordinary differential equation that describes the time evolution of the magnetization for any given initial condition, and it is used to derive both static and dynamic properties of the spin system.

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Although the standard theory of Ising spin systems [1,2] and more generally of spin glasses [3] deals only with thermal equilibrium conditions of these systems, it is quite clear that an understanding of spin dynamics is both desirable and important. Spin dynamics cannot be ignored in experiments and stands out as an important aspect of most computer simulations, and the study of spin dynamics has been undertaken by many people in recent years [4]. In a landmark paper published in 1985, Ogielsky [5] set out very clearly the important issues in the simulation dynamics of spin glasses, and introduced the simple but compelling view that the evolution of the spin configuration can be viewed as a random walk along the edges of a hypercube.

Now let σ_i be the ith spin variable, $P(\sigma,m)$ be the probability that the system assumes the spin configuration $\sigma=\{\sigma_i\}_{i=1,N}$ at the mth time step, and $p(\sigma\rightarrow\sigma')$ be the transition probability from configuration σ to configuration σ' during one time step; in addition assume that there is exactly one spin flip during one time step. Then the following master equation holds [5]:

$$P(\sigma,m+1)=\sum_{I_{\boldsymbol{\sigma}}} p(\sigma'\rightarrow\sigma)P(\sigma',m), \tag{1}$$

where I_σ is the set of the spin configurations adjacent to σ, i.e., the set of all configurations σ' that can be reached from σ with a single spin flip. In an infinite-range Ising spin system with Hamiltonian [6]

$$H=-\frac{J}{N}\sum_{\substack{i,j=1\\ i\neq j}}^{N} \sigma_i\sigma_j-h\sum_i \sigma_i , \tag{2}$$

each state has N neighbors, and the set of neighbors can be divided in two clear-cut subsets with well defined energy and magnetization. Indeed, if a configuration σ has n ''down'' spins and $N-n$ ''up'' spins, the magnetization is $M_n\equiv M(\sigma)=(1/N)(N-2n)$, and the energy is

$$\begin{aligned} E_n\equiv E(\sigma) &= \frac{J}{N}n(N-n)-\frac{J}{2N}n(n-1) \\ &\quad -\frac{J}{2N}(N-n)(N-n-1)-h(N-2n) \qquad (3)\\ &= -\frac{J}{2N}[(N-2n)^2-N]-h(N-2n). \qquad (4) \end{aligned}$$

Because of the very-long-range interactions, the assumed Hamiltonian (2) is not realistic, but—as will be seen later—it gives rise to a very neat phase transition, and has the great advantage of leading to many exact results, both near the critical temperature and away from it.

A single spin flip either raises or lowers the magnetization by $2/N$ and the master equation for magnetization is directly related to the master equation for n:

$$P(n,m+1)=p_-(n+1)P(n+1,m)+p_+(n-1)P(n-1,m), \tag{5}$$

where $p_\pm(n)$ is the probability of raising $(+)$ or lowering $(-)$ n with a single spin flip if the configuration has n ''down'' spins, and $P(n,m)$ is the probability that n spins are ''down'' at the mth time step.

There are $\binom{N}{n}$ ways of selecting the n ''down'' spins in the configuration, therefore—when thermal equilibrium is reached—the probability of finding the system in a configuration with n ''down'' spins is

$$P_n=\binom{N}{n}\frac{e^{-\beta E_n}}{\sum_{n=1,N}\binom{N}{n}e^{-\beta E_n}}, \tag{6}$$

where $P_n=\lim_{m\rightarrow\infty}P(m,n)$, and $\beta=1/kT$ is the usual inverse temperature, and thus for long times Eq. (5) becomes

$$P_n=p_-(n+1)P_{n+1}+p_+(n-1)P_{n-1}, \tag{7}$$

i.e.,

$$\binom{N}{n} e^{-\beta E_n} = p_-(n+1)\binom{N}{n+1} e^{-\beta E_{n+1}} + p_+(n-1) \times \binom{N}{n-1} e^{-\beta E_{n-1}}; \quad (8)$$

these equations—and the condition $p_+(n)+p_-(n)=1$—determine the transition probabilities $p_\pm$. For instance, at high temperature Eq. (8) becomes

$$\binom{N}{n} = p_-(n+1)\binom{N}{n+1} + p_+(n-1)\binom{N}{n-1}, \quad (9)$$

i.e.,

$$1 = p_-(n+1)\frac{N-n}{n+1} + [1-p_-(n-1)]\frac{n}{N-n+1}. \quad (10)$$

It is easy to see that $p_+(n)=n/N$ is a solution of Eq. (10): this is just the probability of choosing at random one of the n ''down'' spins. If we assume that the equilibrium and nonequilibrium transition probabilities are actually the same, we can use the high temperature solution $p_+(n)=n/N$ in Eq. (5) and obtain

$$P(n,m+1) = \frac{n+1}{N}P(n+1,m) + \frac{N-n+1}{N}P(n-1,m). \quad (11)$$

Equation (11) can be easily solved if $N \gg 1$, setting $x=n/N$, $\Delta x = 1/N$, and $t=m\Delta t$ where Δt is the time step, so that $P(n,m) \to P(x,t)$ and

$$P(x,t+\Delta t) = (x+\Delta x)P(x+\Delta x,t) + (1-x+\Delta x)P(x-\Delta x,t); \quad (12)$$

then, after expanding P as a Taylor series and keeping only the first order terms, we find

$$\frac{1}{c}\frac{\partial P}{\partial t} = -(1-2x)\frac{\partial P}{\partial x} + 2P(x,t), \quad (13)$$

where $c=\Delta x/\Delta t$.

The quasilinear partial differential equation (13) can be solved with the *method of characteristics* [7], so that, after introducing a parametric variable s, one obtains the pair of ordinary differential equations (ODE's)

$$\frac{dt}{ds} = \frac{1}{c}, \quad (14)$$

$$\frac{dx}{ds} = 1-2x, \quad (15)$$

which can be easily solved with the initial conditions $t(0)=0$, $x(0)=x_0$; and yield the final result

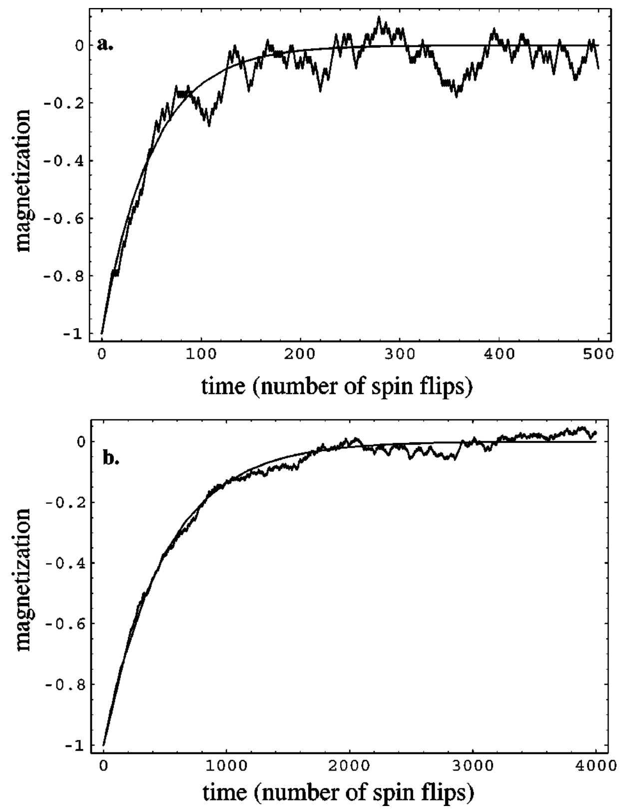

FIG. 1. The jagged curve shows the magnetization in a Monte Carlo simulation of the spin system considered in this paper, at high temperature, while the continuous curve is the plot of the relaxation function (17). The initial magnetization is -1 and the system has (a) 100 spins; (b) 1000 spins; notice that finite-size effects are visibly less important in the large system.

$$x(t) = \frac{1}{2}[1-(1-2x_0)e^{-2ct}] \quad (16)$$

for the variable x and

$$M(t) = M_0 e^{-2ct} \quad (17)$$

for the magnetization, so that this particularly simple case leads to a straightforward exponential relaxation. The method of characteristics also yields an equation for P, $dP/ds = 2P$, which, however, is not relevant in the present context, and can be neglected. Indeed if one starts with a given magnetization, it can be shown that in the large-N limit the magnetization relaxes without spreading, and this is also confirmed by numerical simulations. Figure 1 compares the results of a Monte Carlo simulation of the high temperature dynamics with the magnetization (17), showing good agreement.

At lower temperature we may write $p_+(n)=n/N+f(n)$, and $p_-(n)=1-p_+(n)=(1-n/N)-f(n)$, so that we obtain the equation

$$e^{-\beta E_n}=\left[\frac{N-n}{N}+\frac{N-n}{n+1}f(n+1)\right]e^{-\beta E_{n+1}}+\left[\frac{n}{N}-\frac{n}{N-n+1}f(n-1)\right]e^{-\beta E_{n-1}}, \tag{18}$$

or

$$1=\left[\frac{N-n}{N}+\frac{N-n}{n+1}f(n+1)\right]e^{-\beta(E_{n+1}-E_n)}+\left[\frac{n}{N}-\frac{n}{N-n+1}f(n-1)\right]e^{-\beta(E_{n-1}-E_n)}. \tag{19}$$

The energy E_n can be written in the form $E_n=-(J/2N)[(N-2n)^2-N]-h(N-2n)$, so that $E_{n+1}-E_n=2(J/N)(N-2n-1)+2h$ and $E_{n-1}-E_n=-2(J/N)(N-2n+1)-2h$, and Eq. (19) becomes

$$1=\left[\frac{N-n}{N}+\frac{N-n}{n+1}f(n+1)\right]e^{-2J\beta(N-2n-1)/N-2\beta h}+\left[\frac{n}{N}-\frac{n}{N-n+1}f(n-1)\right]e^{2J\beta(N-2n+1)/N+2\beta h} \tag{20}$$

If we take again the condition $N\gg 1$, and let $x=n/N$, we can solve Eq. (20) and find

$$f(x)=\frac{1-[(1-x)e^{-2J\beta(1-2x)-2\beta h}+xe^{2J\beta(1-2x)+2\beta h}]}{[(1-x)/x]e^{-2J\beta(1-2x)-2\beta h}-[x/(1-x)]e^{2J\beta(1-2x)+2\beta h}}; \tag{21}$$

the function $f(x)$ is plotted in Fig. 2 for different values of the parameters βJ and h/J.

Using the function f given in Eq. (21) we obtain a new quasilinear partial differential equation

$$\frac{1}{c}\frac{\partial P}{\partial t}+[1-2x-2f(x)]\frac{\partial P}{\partial x}=2[1+f'(x)]P(x,t), \tag{22}$$

and then, using again the method of characteristics, we find the pair of ODE's

$$\frac{dt}{ds}=\frac{1}{c}, \tag{23}$$

$$\frac{dx}{ds}=1-2x-f(x), \tag{24}$$

i.e.,

$$\frac{1}{c}\frac{dx}{dt}=1-2x-f(x). \tag{25}$$

Equation (25) describes the thermodynamic properties of the system as well as its dynamics, because the system considered here is dissipative and therefore for long times the derivative (25) must vanish, i.e., the following system of equations must be satisfied:

$$y=1-2x, \tag{26a}$$

$$y=f(x), \tag{26b}$$

and it can be solved numerically [see Fig. 3(a)]. In the symmetrical $h=0$ case, there is just one solution if $f'(1/2)>-2$, and there are three solutions if $f'(1/2)<-2$, and since $f(1/2)=0$, after a straightforward evaluation of the derivative of f, we find that the system bifurcates at $\beta J=2$, i.e., the critical temperature with zero external field is $T_c(0)=J/2k$.

Now we notice that near $x=1/2$ (which corresponds to a null magnetization)

$$\begin{aligned}\frac{1}{c}\frac{dx}{dt}&\approx 1-2x-f'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)\\&=1-2x+\beta J\left(x-\frac{1}{2}\right)\\&=1-2x+2\frac{T_c(0)}{T}\left(x-\frac{1}{2}\right)\end{aligned} \tag{27}$$

or

$$\frac{1}{c}\frac{dM}{dt}\approx -2\left(1-\frac{T_c(0)}{T}\right)M, \tag{28}$$

so that

$$M(t)\approx M(0)e^{-2c\{[T-T_c(0)]/T\}t}. \tag{29}$$

This last formula is not adequate near the critical temperature; Eq. (27) must be approximated with the inclusion of $f'''(x)$ (the second derivative vanishes at $x=1/2$):

$$\frac{1}{c}\frac{dx}{dt}\approx 1-2x-f'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)-\frac{1}{6}f'''\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^3, \tag{30}$$

and the equation for the magnetization becomes

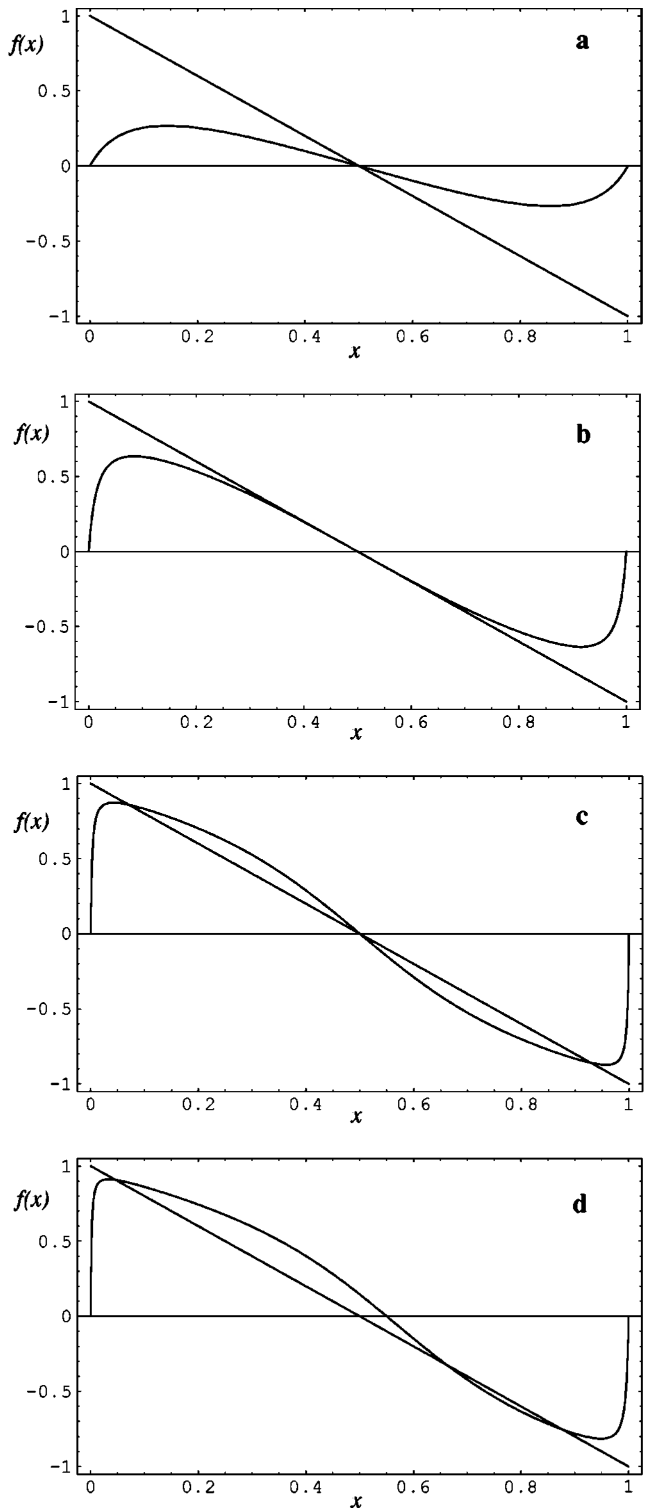

FIG. 2. Plot of the function $f(x)$ vs $x=n/N$ for different values βJ and $h=0$. (a) $\beta J=1$; (b) $\beta J=2$; (c) $\beta J=3$. The last figure shows a plot with nonzero field: (d) $\beta J=3$ and $h/J=0.1$. The straight line in each figure is the function $1-2x$, and the intersections of $f(x)$ with the straight line are the solutions of the system of equations (26).

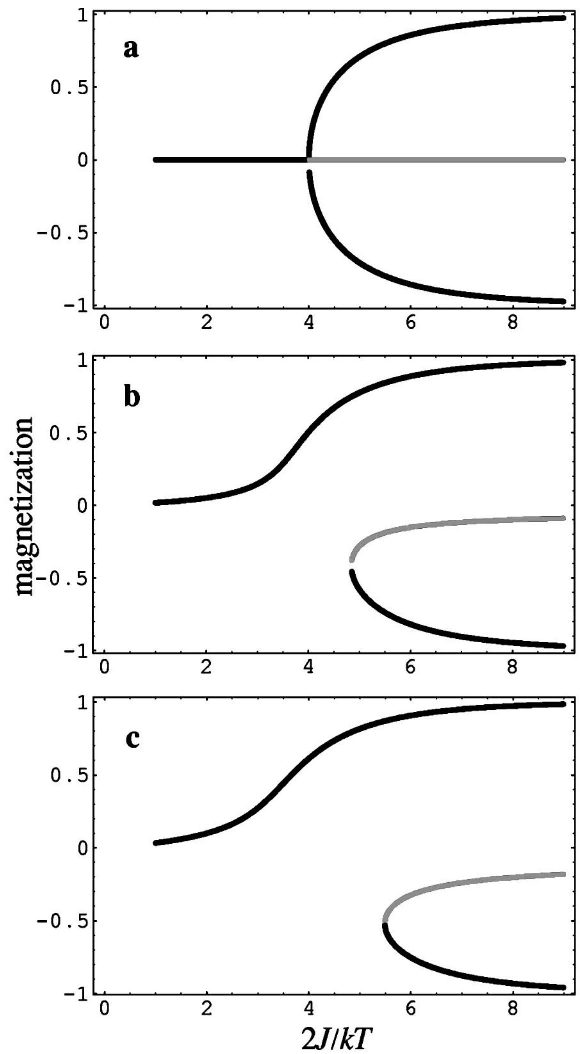

FIG. 3. Numerical solutions of the system of equations (26) for different values of h/J. (a) Solution for $h=0$: the system displays a supercritical pitchfork bifurcation as the parameter $2\beta J$ becomes larger than 4: the attractors of the system are shown in black, the repellor in gray. (b) and (c) ($h/J=0.05$ and $h/J=0.1$): when $h\neq 0$ the original pitchfork bifurcation becomes a supercritical saddle-node bifurcation.

$$-\frac{1}{2c}\frac{dM}{dt}=\left(1-\frac{T_c(0)}{T}\right)M+\left(1-2\frac{T_c(0)}{T}+\frac{4}{3}\frac{T_c^2(0)}{T^2}\right)M^3. \tag{31}$$

In a narrow region about $T_c(0)$ and $M\approx 0$, where

$$\left(1-\frac{T_c(0)}{T}\right)\lesssim\left(1-2\frac{T_c(0)}{T}+\frac{4}{3}\frac{T_c^2(0)}{T^2}\right)M^2, \tag{32}$$

the cubic term becomes dominant, and

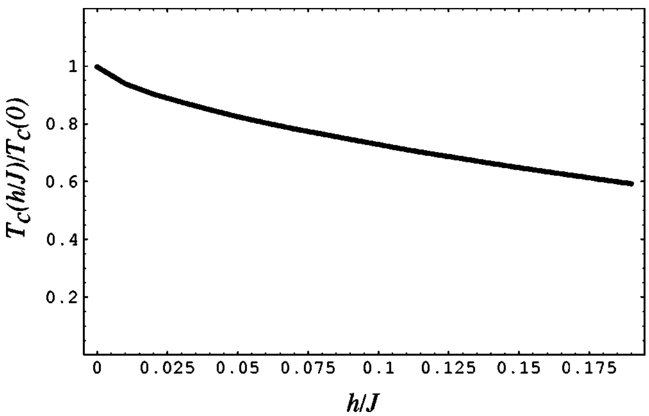

FIG. 4. Plot of the critical temperature $T_c(h/J)$ vs h/J [the critical temperature $T_c(h/J)$ is the temperature at which the solution of Eq. (26) starts bifurcating].

$$M(t)\approx \pm\left(\frac{1}{M^2(0)}+\frac{4}{3}ct\right)^{-1/2}, \tag{33}$$

so that in this region the relaxation is no longer exponential.

These calculations can be easily extended to the case $h\neq 0$: Figs. 3(b) and 3(c) show the resulting supercritical saddle-node bifurcation [8] at two different values of the h/J ratio, and Fig. 4 shows $T_c(h/J)$ vs h/J: now there is one ''stable'' magnetization, parallel to the external field, and an antiparallel ''metastable'' magnetization.

Notice that the number of ''down'' spins can be directly computed from the magnetization M, $n=N(1-M)/2$, and that this means that the internal energy per spin can be written as a function of the magnetization $M(T,h/J)$:

$$\frac{1}{N}E(T,h/J)=-J\left(\frac{1}{2}M^2(T,h/J)+\frac{h}{J}M(T,h/J)\right)+\frac{J}{2N}; \tag{34}$$

then the heat capacity per spin $(1/N)C(T,h/J)$ at different values of the h/J ratio can be computed from the numerical solutions of $M(T,h/J)$, such as those shown in Fig. 3, using the formula

$$\frac{1}{N}C(T,h/J)=\frac{1}{N}\left.\frac{\partial E}{\partial T}\right|_{h/J}=\frac{1}{N}\left.\frac{\partial E}{\partial M}\right|_{h/J}\left.\frac{\partial M}{\partial T}\right|_{h/J} \tag{35}$$

$$=-J\left(M(T,h/J)+\frac{h}{J}\right)\left.\frac{\partial M}{\partial T}\right|_{h/J}; \tag{36}$$

Fig. 5 shows plots of $C(T,h/J)/kN$ at different values of the h/J ratio, both for the ''stable'' and for the ''metastable'' magnetization.

When there is a nonzero external magnetic field the f function is no longer symmetric [see Fig. 1(d)], but it is still true that at the critical temperature $T_c(h/J)$ the following equations must hold at x_c, the value of x at which $1-2x$ is tangent to $f(x)$:

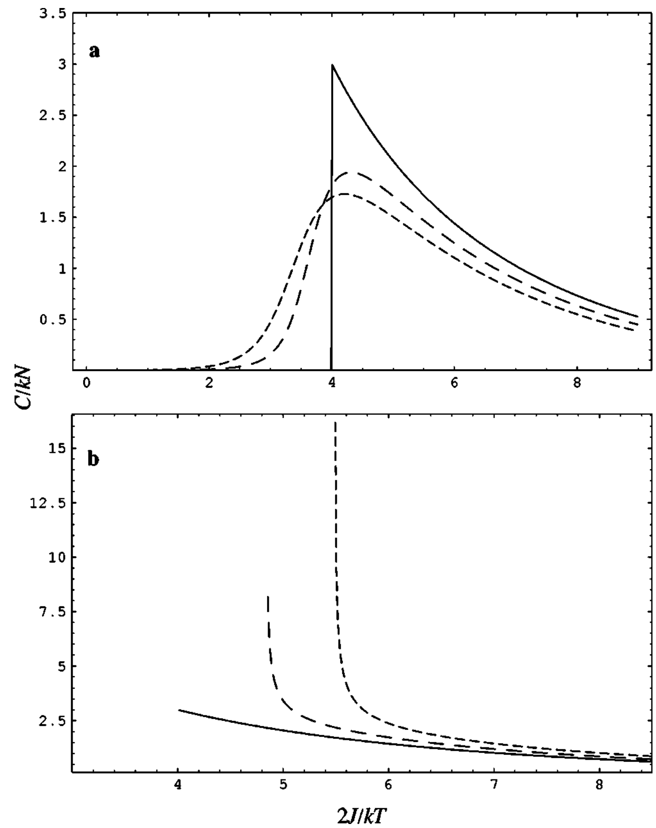

FIG. 5. Plot of the heat capacity per spin for different values of h/J: $h=0$ (solid line); $h/J=0.05$ (long dashes); $h/J=0.1$ (short dashes). (a) shows the heat capacity of the stable state, while (b) shows the heat capacity of the metastable state.

$$f(x_c)=1-2x_c, \tag{37}$$

$$f'(x_c)=-2 \tag{38}$$

(when $h=0$, $x_c=1/2$). Thus near the critical temperature Eq. (25) becomes (for $h\neq 0$)

$$\frac{1}{c}\frac{dx}{dt}\approx 1-2x-(1-2x_c)-f'(x_c)(x-x_c)-\frac{1}{2}f''(x_c)(x-x_c)^2 \tag{39}$$

$$=-\frac{1}{2}f''(x_c)(x-x_c)^2. \tag{40}$$

This means that in the case $h\neq 0$ also the relaxation is nonexponential near the critical temperature, but now the exponent is different, it is -1 rather than $-1/2$ as before.

In this paper I have found an exact solution for a particular Ising spin model, with a method that is different from the established ones: it is natural to wonder where the difference from those other methods lies. Obviously the static properties might have been calculated from the partition function

[6], which is easily written down because all states with a given magnetization share the same energy,

$$Z=\sum_{n=0}^{n=N}\binom{N}{n}e^{-\beta E_n}$$
$$=\sum_{n=0}^{n=N}\binom{N}{n}e^{(J\beta/2N)[(N-2n)^2-N]+\beta h(N-2n)}; \qquad (41)$$

however, this partition function tells us nothing about the dynamics. In addition the treatment exposed here also provides a clear connection between a simple spin system and the theory of dynamical systems and bifurcations.

I wish to thank Professor G. Careri for having brought to my attention a paper by Campbell [9], which introduced me to the problems of spin dynamics, and for many stimulating discussions.

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[4] There is a large body of literature on spin dynamics and it is impossible to write a short list without being unfair to many contributors; here I mention only the seminal paper by R. J. Glauber [J. Math. Phys. **4**, 294 (1963)] and a recent paper by J.-Y. Zhu and Z. R. Yang [Phys. Rev. E **61**, 210 (2000)], which can be used as an entry point to the vast existing literature. See also the paper by Ogielsky [5].
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